The Slow-Roll and Rapid-Roll Conditions in The Space-like Vector Field Scenario

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Abstract

In this note we derive the slow-roll and rapid-roll conditions for the minimally and non-minimally coupled space-like vector fields. The function $f(B^2)$ represents the non-minimal coupling effect between vector fields and gravity, the $f = 0$ case is the minimal coupling case. For a clear comparison with scalar field, we define a new function $F = \pm B^2/12 + f(B^2)$ where $B^2 = A_\mu A^\mu$, A_μ is the "comoving" vector field. With reference to the slow-roll and rapid-roll conditions, we find the small-field model is more suitable than the large-field model in the minimally coupled vector field case. And as a non-minimal coupling example, the $F = 0$ case just has the same slow-roll conditions as the scalar fields.

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1 Introduction

Inflation, as a theory to describe the period of acceleration in the early universe, was introduced as a way to solve the problems in the standard big-bang theory [1, 2]. Dark energy, as a candidate for the period of acceleration in the late universe, was suggested by a combination of different cosmic probes that primarily involves Supernova data [3, 4]. The scalar field is the most popular candidate for the dynamical source of the two accelerations. We could assume the scalar field isotropic and homogeneous naturally. The assumption that the scalar field is homogeneous and isotropic coincides with the observable isotropic and homogeneous Friedmann-Robertson-Walker (FRW) background automatically. However, the fundamental scalar field, has not been probed until now. On the contrary, the vector field is common in the realistic world. The vector fields inflation scenario was proposed by Ref. [5, 6], recently extended to higher spin field [7, 8, 9].

The vector field $A_\mu = (A_0, A_1, A_2, A_3)$ is anisotropic in nature for the oriented components. To coincide with the observable isotropic and homogenous FRW background, there are three models which give out isotropy in vector field scenarios. The first one is to the vector fields $A_\mu = (A_0, 0, 0, 0)$ with only the temporal component which is isotropic obviously [10, 11, 12, 13, 14, 15, 16, 17, 18]. The second model, called "cosmic triad", has three spatial components equal to each other and orthogonal to each other in which the vector field has such a form $A_\mu = (0, A, A, A)$ [19, 20, 21](see also [22, 23, 24, 25, 26, 27] for exact isotropic solutions of the Einstein-Yang-Mills system based on the same idea). And the third scenario, called "N-flation" vector scenario, has a large number of randomly oriented fields [28, 29, 30] in which the vector field has a form as $A_\mu = (0, A_1, A_2, A_3)$. Under certain approximations, the forms of last two space-like scenarios are very similar to each other, which play the leading role in this note.

However, even after the isotropic problem is solved, the slow-roll problem needs to be solved to make the duration of inflation last long enough in vector field scenarios. There are two questions about the slow-roll problem: one is what's the exact form of the slow-roll conditions in the vector field scenarios; the other is whether the de-Sitter phase will appear or not without slow rolling vector field. Whatever, the dark energy dominating acceleration phase only requires one e-folding number by observations. Even the vector field driven acceleration could not offer the e-folding number as large as 60 for inflation, a period of acceleration before or after the main part of inflation can also alleviate some cosmic problems (such as the moduli problem) [31]. It seems that not only the slow-roll

conditions, but also the rapid-roll conditions which is looser proposed by Ref. [32, 33, 34] are worthy of considering. Because under rapid-roll conditions the universe can get a de-Sitter phase as well. Moreover, based on the non-minimal coupling effect of changing dynamics of vector field, we will consider the non-minimal coupling term in the discussions.

In the following section, a concise introduction will be given out for "cosmic triad" and "N-flation" vector field scenarios. They can be expressed in a similar form with the non-minimally coupled scalar field. Then, the non-minimally coupled vector field scenario is given out as well. In Sec. 3 and 4, the slow-roll and rapid-roll conditions both in the minimal and non-minimal coupling cases will be discussed as our main aim. Special examples will be given out during the descriptions.

2   Space-like Vector Field Scenarios

To make a complete description, it is appropriate to consider the possibility of $w = p/\rho < -1$ [20, 35] which is suggested by dark energy observations [36]. The discussions on both the positive and negative kinetic energy cases in the vector field scenario are included in the note. Although the negative kinetic energy case may have a lot of theoretical problems, it may be phenomenologically significant and worth putting other theoretical difficulties aside temporally.

2.1   "Cosmic Triad" Vector Field Scenario

The "cosmic triad" vector field scenario [20] composes by a set of three identical self-interacting vectors which could naturally arise (for instance from a gauge theory with $SU(2)$ or $SO(3)$ gauge group). The three vector fields, which are minimally coupled with gravity, have the action

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \sum_{a=1}^{3} \left(\pm \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + V(A^{a2}) \right) \right], \tag{1}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ and $A^{a2} = g^{\mu\nu} A^a_\mu A^a_\nu$, Latin indices label the different fields $(a, b, ... = 1...3)$, and Greek indices label the different space-time components $(\mu, \nu, ... = 0...3)$. The term $\pm F^a_{\mu\nu} F^{a\mu\nu}/4$ could be considered as the Maxwell type kinetic energy term, the case with a sign "$+$" corresponds to the positive kinetic energy term and the case with

a sign " $-$ " corresponds to the negative kinetic energy case in the vector field scenario. The term $V(A^2)$ is the potential of the vector field. The four-dimension homogenous and isotropic FRW metric has such a form

$$ds^2 = -dt^2 + a^2(t)\delta_{ik}dx^i dx^k, \tag{2}$$

where a is the scale factor, we consider the spatial curvature $k = 0$. And we could define a new variable called "physical" vector field B_i as discussed in Ref. [20]

$$B_i = \frac{A_i}{a} = aA^i, \tag{3}$$

where A_i is called "comoving" vector field. The related equation $B^2 = B_i B_i = A_\mu A^\mu = A^2$ could be conveniently gotten in the FRW background. Then we will express most equations in term of B_i and B^2 in the following discussions.

By varying the action in Eq. (1) with respect to $g_{\mu\nu}$, the (00) and (ii) components in the Einstein equations, called Friedmann and Raychaudhuri equation, can be obtained

$$H^2 = \frac{8\pi G}{3}\rho, \tag{4}$$

$$\dot{H} = -4\pi G(\rho + p), \tag{5}$$

where $H = \dot{a}/a$ is the Hubble parameter. The ansatz in "cosmic triad", that the three vectors are equal and orthogonal to each other, can be expressed as

$$A_\mu^b = \delta_\mu^b B(t) \cdot a. \tag{6}$$

The corresponding energy density ρ and pressure p are given by

$$\rho = \pm\frac{3}{2}(\dot{B}_i + HB_i)^2 + 3V(B^2), \tag{7}$$

$$p = \pm\frac{1}{2}(\dot{B}_i + HB_i)^2 - 3V(B^2) + V_i'B_i, \tag{8}$$

where the dot means a derivative with respect to time t, and the prime with an index i denote a derivative with respect to vector field B_i, for example $V_i' = dV/dB_i$, two primes used in the following calculations are given by $'' = d^2/dB_i^2 = d^2/dB^2$. And equations of motion of vector field could be obtained by varying the action in Eq. (1) with respect to vector fields A_μ^a

$$\ddot{B}_i + 3H\dot{B}_i + (2H^2 + \dot{H})B_i \pm V_i' = 0. \tag{9}$$

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In the above equations, even when $V = constant$, the term $(2H^2 + \dot{H})B_i$ can make B_i evolve as if there were an additional effective potential $(2H^2 + \dot{H})B^2/2$.

For a clear comparison with scalar field, we define a function $F(B^2) = \pm B^2/12$. The forms of energy density, pressure and equations of motion in vector field become

$$\rho = 3\left(\pm\frac{1}{2}\dot{B}_i^2 + V(B^2) + 6H(\dot{F} + HF)\right), \tag{10}$$

$$p = 3\left(\pm\frac{1}{2}\dot{B}_i^2 - V(B^2) - 2\ddot{F} - 4H\dot{F} - 2F(2\dot{H} + 3H^2)\right), \tag{11}$$

$$\ddot{B}_i + 3H\dot{B}_i \pm 6F_i'(2H^2 + \dot{H}) \pm V_i' = 0, \tag{12}$$

which are very similar to those in the non-minimally coupled scalar field. The latter has a action like

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + \frac{(\nabla\phi)^2}{2} - V(\phi) - f(\phi)R\right), \tag{13}$$

where $f(\phi)$ denotes the non-minimal coupling between the field and gravity. Then the forms of energy density, pressure and equation of motion for scalar field are

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi^2) + 6H(\dot{f} + Hf), \tag{14}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi^2) - 2\ddot{f} - 4H\dot{f} - 2f(2\dot{H} + 2H^2), \tag{15}$$

$$\ddot{\phi} + 3H\dot{\phi} + V' + 6f'(\dot{H} + 2H^2) = 0. \tag{16}$$

where the prime in Eq. (16) is a derivative with respect to scalar field ϕ. The differences between Eq. (10) and Eq. (14), and Eq. (11) and Eq. (15) are only in the coefficients. And the equations of motion are nearly the same.

2.2 "N-flation" Vector Field Scenario

The "N-flation" vector field scenario [28], inspired by the "N-flation" scalar field model [37], has N randomly oriented vector fields. Following the assumptions in Ref. [28], all the vector fields have equal potentials and same orders of initial values. In "N-flation" vector field scenario, although the anisotropy can be counterbalanced by the randomly oriented fields mainly, the universe is slightly anisotropic. Concretely speaking, until the end of inflation, the vector fields will remain an anisotropy of order $1/\sqrt{N}$. As long as N is large

enough, the anisotropy could be ignored [28]. The equation

$$\sum_{a=1}^{N} B_i^a B_j^a \simeq \frac{N}{3} B^2 \delta_j^i + \mathcal{O}(1)\sqrt{N} B^2, \tag{17}$$

notes that the universe can be treated as if it was isotropic after assuming $B^2 < 3m_{pl}^2/\sqrt{N}$ [28]. An isotropic universe is implied in the following discussion.

The "N-flation" vector field scenario has such an action

$$S = \int d^4x \sqrt{-g}[\frac{R}{16\pi G} - N(\pm\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(B^2))]. \tag{18}$$

With an additional function $F = \pm B^2/12$, the energy density, pressure and the equations of motion under these assumptions can be simplified as

$$\rho = N \left(\pm\frac{1}{2}\dot{B}_i^2 + V(B^2) + 6H(\dot{F} + HF) \right), \tag{19}$$

$$p = N \left(\pm\frac{1}{2}\dot{B}_i^2 - V(B^2) - 2\ddot{F} - 4H\dot{F} - 2F(2\dot{H} + 3H^2) \right), \tag{20}$$

$$\ddot{B}_i + 3H\dot{B}_i \pm 6F_i'(2H^2 + \dot{H}) \pm V_i' = 0. \tag{21}$$

The above forms in "N-flation" are just the same as those in the "cosmic triad" scenario. The energy density in Eqs. (14) and (19)and the pressure in Eqs. (15) and (20) are different only in the coefficients which is characterized by the number of the fields. Therefore in the following we will only use Eqs. (19), (20) and (21). The $N = 3$ case is regarded as the "cosmic triad" vector field scenario and the large N case is regarded as the "N-flation" vector field scenario. And based on these similarities, we will extend minimal coupling to non-minimal coupling for vector fields in the following discussions.

2.3   Non-Minimal Coupling Vector Field Case

In the vector field scenario, the non-minimal coupling term is used to satisfy the slow-roll conditions. Without non-minimal coupling, the vector field could only be used as curvaton [9, 38, 39, 40]. To give a complete examination of the rolling, the possible non-minimal coupling between the vector field and gravity is also included. Let us start from such an action

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + N \left(\pm\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(A^2) - f(A^2)R \right) \right], \tag{22}$$

where the function $f(A^2)$ shows the non-minimal coupling effect, and A^2 also can be rewritten as B^2 by Eq. (3). Under "cosmic triad" and "N-flation" vector field assumptions, the Friedmann and Raychaudhuri equations remain the same, but the energy density, pressure and the equations of motion are modified to

$$\rho \simeq N\left(\pm\frac{1}{2}(\dot{B}_i + HB_i)^2 + V + 6H(\dot{f} + Hf)\right), \tag{23}$$

$$p \simeq N\left(\pm\frac{1}{6}(\dot{B}_i + HB_i)^2 - V + \frac{V_i'B_i}{3} - 2\ddot{f} - 4H\dot{f} - 2f(2\dot{H} + 3H^2)\right), \tag{24}$$

$$\ddot{B}_i + 3H\dot{B}_i \pm V_i' + (2H^2 + \dot{H})B_i + 6f_i'(\dot{H} + 2H^2) = 0, \tag{25}$$

where the non-minimal coupling parameter f is the additional variable. After redefining the function $F = f(B^2) \pm B^2/12$, the above equations can be reexpressed in a definite form

$$\rho \simeq N\left(\pm\frac{1}{2}\dot{B}_i^2 + V + 6H(\dot{F} + HF)\right), \tag{26}$$

$$p \simeq N\left(\pm\frac{1}{2}\dot{B}_i^2 - V - 2\ddot{F} - 4H\dot{F} - 2F(2\dot{H} + 3H^2)\right), \tag{27}$$

$$\ddot{B}_i + 3H\dot{B}_i + 6F_i'(\dot{H} + 2H^2) \pm V_i' = 0, \tag{28}$$

which can be analyzed in the same as the minimal coupling case. The latter corresponds to $f = 0$ ($F = \pm B^2/12$). For convenience, in the following section we first discuss the non-minimally coupled vector field case, then apply the results to the minimal coupling case.

Especially, the $F = 0$ case corresponds to the vector field inflation discussed in Ref. [28]; furthermore according to the equations

$$\rho \simeq N(\frac{1}{2}\dot{B}_i^2 + V), \tag{29}$$

$$p \simeq N(\frac{1}{2}\dot{B}_i^2 - V), \tag{30}$$

the energy density and the pressure are nearly the same as those in the minimally coupled scalar fields. And as Ref. [28] argued, the slow-roll conditions can be realized in this non-minimal coupling case. The arguments above indicate that the behaviors of the vector fields in minimal and non-minimal coupling case are totally different from those of the scalar fields. So it is necessary to investigate the vector field slow-roll conditions specially.

3 Slow-Roll Conditions in the Non-minimally Coupled Vector Field

The reason, that inflation needs slow-roll conditions, is the Hubble parameter H should be nearly constant and the universe should be in de-Sitter phase for a long period of time. Combined with Eqs. (26), (27) and (28), by defining a function $\Omega = 1 - 2F/m_{pl}^2$, Friedmann and Raychaudhuri equations for non-minimally coupled vector field become

$$H^2\Omega + H\dot{\Omega} = \frac{N}{3m_{pl}^2}(\frac{1}{2}\dot{B}^2 + V), \tag{31}$$

$$\ddot{\Omega} - H\dot{\Omega} + 2\dot{H}\Omega = -\frac{N\dot{B}^2}{3m_{pl}^2}, \tag{32}$$

and the equations of motion (9) could be rewritten as

$$\ddot{B}_i + 3H\dot{B}_i \pm V_i' \mp 3m_{pl}^2\Omega_i'(\dot{H} + 2H^2) = 0, \tag{33}$$

where $m_{pl}^{-2} = 8\pi G$ is the planck mass. A long period of inflation requires the potential dominates the evolution of the universe, and the slow varying of the field means the acceleration of the field should be neglected. The conditions $\dot{B}_i^2 \ll V$, $|\dot{\Omega}| \ll H\Omega$, $\ddot{B}_i \ll H|\dot{B}_i|$ and $|\ddot{B}_i| \ll |V_i'|$ could reduce to

$$H^2\Omega \simeq \frac{NV}{3m_{pl}^2}, \tag{34}$$

$$3H\dot{B}_i \simeq -\widetilde{V}_i', \tag{35}$$

where $\widetilde{V}_i' = \pm V_i' \mp 6m_{pl}^2\Omega_i'H^2 \mp 3m_{pl}^2\Omega_i'\dot{H}$, $\widetilde{V}_i$ could be regarded as the effective potential. And we could define three parameter related to slow-roll process

$$\epsilon \equiv \frac{m_{pl}^2\Omega\widetilde{V}_i'^2}{2NV^2}, \tag{36}$$

$$\eta \equiv \frac{m_{pl}^2\Omega\widetilde{V}''}{NV}, \tag{37}$$

$$\delta \equiv \frac{m_{pl}^2\Omega_i'\widetilde{V}_i'}{NV} = \frac{-2F_i'\widetilde{V}_i'}{V} \tag{38}$$

for preparation. And for theoretical consistency, we need to know the form of slow-roll conditions in detail that satisfy the equations $\dot{B}_i^2 \ll V$, $|\dot{\Omega}| \ll H\Omega$, $\ddot{B}_i \ll H|\dot{B}_i|$ and

$|\ddot{B}_i| \ll |V'_i|$. Varying Eq. (35) and using Eq. (34), we can get

$$\frac{\dot{B}_i^2}{V} \simeq \frac{m_{pl}^2 \Omega \widetilde{V}_i'^2}{3NV^2} = \frac{2}{3}\epsilon, \tag{39}$$

$$\frac{\dot{\Omega}}{H\Omega} \simeq -\frac{m_{pl}^2 \Omega_i' \widetilde{V}_i'}{NV} = -\delta, \tag{40}$$

$$\frac{\ddot{B}_i}{H\dot{B}_i} \simeq -\frac{\dot{H}}{H^2} - \frac{m_{pl}^2 \Omega \widetilde{V}''}{NV} = -\frac{\dot{H}}{H^2} - \eta, \tag{41}$$

$$\frac{\ddot{B}}{V_i'} \simeq \frac{\dot{H}}{3H^2}\frac{\widetilde{V}_i'}{V_i'} + \frac{m_{pl}^2 \Omega \widetilde{V}''}{3NV}\frac{\widetilde{V}_i'}{V_i'} = \frac{\widetilde{V}_i'}{3V_i'}(\frac{\dot{H}}{H^2} + \eta), \tag{42}$$

where $'' = d^2/dB_i^2 = d^2/dB^2$. By requiring $|\epsilon| \ll 1$ and $|\delta| \ll 1$, $\dot{B}_i^2 \ll V$ and $|\dot{\Omega}| \ll H\Omega$ could be satisfied. And if we neglect $\dot{\Omega}$ based on the definition of δ, Eq. (32) can be rewritten as $-(H\Omega)' + 3\dot{H}\Omega = -N\dot{B}_i^2/3m_{pl}^2$. Combining it with Eq. (34), we obtain

$$\frac{\dot{H}}{H^2} = \frac{\dot{\Omega}}{2H\Omega} - \frac{m_{pl}^2 \Omega \widetilde{V}'^2}{6NV^2} = -\frac{\delta}{2} - \frac{\epsilon}{3}, \tag{43}$$

and we turns into $\dot{H}/H^2 \ll 1$ in the condition $|\epsilon| \ll 1$ and $|\delta| \ll 1$. Moreover, adding the condition $|\eta| \ll 1$, we can get $\ddot{B}_i \ll H|\dot{B}_i|$ and $|\ddot{B}_i| \ll |V'_i|$. In brief, $|\epsilon|, |\delta|, |\eta| \ll 1$ could be given out as slow-roll conditions ground on the above discussions. Compared to the minimally coupled scalar field case with only two slow-roll parameters (ϵ and η), the additional slow-roll parameter δ in the vector field scenario shows a constraint on the function Ω, in other word, on the function F, which is not zero but $\pm B^2/2$ even in the minimally coupled vector field scenario.

When $F = 0$, an special example of non-minimal coupling case presented in Tab.(1), we can get $\Omega = 1$ and $\widetilde{V}_i' \simeq V_i'$. And now the definition gives the parameter $\delta = 0$ directly, so δ can be ignored. Furthermore, by putting $\Omega = 1$ and $\widetilde{V}_i' \simeq V_i'$ into slow-roll conditions, we can get $\epsilon = m_{pl}^2 V_i'^2/2V^2$ and $\eta = m_{pl}^2 V''/V$ which are the same as the definitions of the standard slow-roll parameters in the minimally coupled scalar fields. It indicated that this special case may have the same good property as the scalar field scenario.

Table 1: Two cases.

Variable	Minimal coupling case	An special example[28]
$f(B^2)$	$f = 0$	$f = \mp B^2/2$
$F = \pm B^2/2 + f(B^2)$	$F = \pm B^2/2$	$F = 0$

4 Rapid-Roll Conditions in the Non-minimally Coupled Vector Field

As presented in the introduction, getting a de-Sitter phase is critical in the early and late period of acceleration, which demands $\dot{H}/H^2 \ll 1$. As Eq. (43) noticed, $\dot{H}/H^2 \ll 1$ only requires $\epsilon, \delta \ll 1$. Hence, if we only ask for a de-Sitter phase without considering how long it would last, the slow-roll conditions could be relaxed, especially the condition related to η parameter. That is the motivation of rapid-roll inflation [32, 33, 34], which could be regarded as a new model of fast-roll inflation [31]. The rapid-roll inflation can lead a period of acceleration (de-Sitter phase) as well, but requires looser conditions compared to slow-roll inflation.

However, the scalar field rapid-roll inflation is a novel type of inflation with a non-minimal coupling term in which inflation could occur without slow-rolling field [33]. Here, because the similarity between the equations in the minimally and non-minimally coupled scalar field scenario, the minimally coupled vector field could induce rapid-roll inflation as well. We will define the vector field rapid-roll inflation by following Ref. [33], which starts with the Friedmann Equation and the equations of motion

$$H^2 = \frac{N}{3m_{pl}^2}\left(\pm\frac{1}{2}\dot{B}_i^2 + V + 6H(\dot{F} + HF)\right),$$ (44)

$$\ddot{B}_i + 3H\dot{B}_i \pm V_i' \pm 6F_i'(\dot{H} + 2H^2) = 0.$$ (45)

After defining a new available viable $\vartheta_i = \dot{B}_i \pm 6HF_i'$, the above equations can be rewritten as

$$H^2 = \frac{N}{3m_{pl}^2}\left(\pm\frac{1}{2}\vartheta_i^2 + V + 6H(\dot{F} - 3F'^2)\right),$$ (46)

$$\dot{\vartheta}_i + 2H\vartheta_i \pm V_i' + (1 \mp 6F'')H\dot{B}_i = 0.$$ (47)

It is still need to assume that the potentials of vector fields dominate over the kinetic energy term in the rapid-roll inflation. Meanwhile, we provides a dimensionless parameter

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c to loose the rolling of vector fields. That means Eqs. (46) and (47) should be simplified to

$$H^2 \simeq \frac{NV}{3m_{pl}^2},$$
(48)

$$(c+2)H\vartheta_i \simeq -V_i'.$$
(49)

And we could define three parameters as following for preparation:

$$\epsilon_c \equiv \pm\frac{m_{pl}^2 V_i'^2}{2NV^2} + \frac{2N(c+2)^2}{3m_{pl}^2}(F - 3F_i'^2),$$
(50)

$$\eta_c \equiv \frac{m_{pl}^2 V''}{NV} + \frac{2(c+2)F_i'V''}{V_i'} + \frac{c(c+2)}{3} - \frac{c+2}{3}(1 \mp 6F'') \mp \frac{2N(c+2)^2}{3}\frac{(1 \mp 6F'')F_i'V}{m_{pl}^2 V_i'}$$
(51)

$$\delta_c \equiv \frac{m_{pl}^2 V_i'^2}{2N(c+2)V^2} \pm \frac{F_i'V_i'}{V}.$$
(52)

To satisfy Eqs. (48) and (49), we needs the following inequalities

$$\frac{\pm\vartheta_i^2/2 + 6H^2(F - 3F_i'^2)}{V} \simeq \frac{\pm 3m_{pl}^2 V_i'^2}{2N(c+2)^2 V^2} + \frac{2N}{m_{pl}^2}(F - 3F_i'^2) = \frac{3}{(c+2)^2}\epsilon_c \ll 1, \quad (53)$$

$$\frac{\dot\vartheta_i - cH\vartheta_i + H(1 \mp 6F'')(\vartheta_i \mp 6HF_i')}{(c+2)H\vartheta_i}$$

$$\simeq -\frac{\dot H}{(c+2)H^2} - \frac{3m_{pl}^2 V''}{NV(c+2)^2} - \frac{6F_i'V''}{(c+2)V_i'} - \frac{c}{c+2} + \frac{1 - 6F''}{c+2} \pm \frac{2(1 \mp 6F'')F_i'V}{m_{pl}^2 V_i'}$$

$$= -\frac{\dot H}{(c+2)H^2} - \frac{3}{(c+2)^2}\eta_c \ll 1.$$
(54)

Eq. (54) require $\dot H/H^2 \ll 1$, the de-Sitter phase as well. So, for the consideration of consistence, by using Eq. (48) we get

$$\left|\frac{\dot H}{H^2}\right| = \left|\frac{-3m_{pl}^2 V_i'^2}{2N(c+2)V^2} \mp \frac{3V_i'F_i'}{V}\right| = |-3\delta_c|.$$
(55)

The above equation notes after requiring $|\epsilon_c|, |\eta_c|, |\delta_c| \ll 1$, Eqs. (48) and (49) could be consistently satisfied.

The value of c is determined by η_c, as the rapid-roll conditions require $\eta_c \simeq 0$, we could get

$$c = -2 - \frac{x}{2y} \pm^* \sqrt{(\frac{x}{2y})^2 - \frac{m_{pl}^2 V''}{yNV}},$$
(56)

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where the sign "$\pm^*$" suggests two different solutions, $x = 2F_i'V''/V_i' - 2/3 - (1 \mp 6F'')/3$ and $y = \mp 2NF_i'V(1 \mp 6F'')/3m_{pl}^2V_i' + 1/3$.

Take $F = 0$ for example, the rapid-roll conditions can be reduced to

$$\epsilon_c = \pm \frac{m_{pl}^2 V_i'^2}{2NV^2} \ll 1, \tag{57}$$

$$\eta_c = \frac{m_{pl}^2 V''}{NV} + \frac{(c-1)(c+2)}{3} \ll 1, \tag{58}$$

$$\delta_c = \frac{m_{pl}^2 V_i'^2}{2N(c+2)V^2} \ll 1. \tag{59}$$

If $\eta_c \simeq 0$,

$$c = -\frac{1}{2} \pm^* \sqrt{\frac{9}{4} - \frac{3m_{pl}^2 V''}{NV}}. \tag{60}$$

The differences between ϵ_c and δ_c are in the coefficients, so either is enough. Particularly speaking, the number of the slow-roll parameters could be reduced to two. And when $V'' \ll V$, $c = 1$, the form of η_c ($\eta_c \simeq m_{pl}^2 V''/V$) is similar to the η parameter definition in the scalar field. Therefore, when $V'' \ll V$, the rapid-roll vector field inflation coincides with the slow-roll vector field case.

5 Minimally Coupled Vector Field

5.1 Slow-Roll and Rapid-Roll Conditions

As Tab.(1) notes, the minimally coupled vector field case corresponds to $F = \pm B^2/12$. From the minimal coupling view, we can get $\Omega = 1 \mp B^2/(6m_{pl}^2)$, $\Omega_i' = \mp B_i/(3m_{pl}^2)$ and $\widetilde{V}_i' \simeq \pm V_i' + 2H^2 B_i$. Using these conditions, the slow-roll conditions could be obtained

$$\epsilon = \frac{\Omega}{2N}\left(\frac{m_{pl}V_i'}{V} + \frac{2NB_i}{3m_{pl}}\frac{1}{1 - B^2/6m_{pl}^2}\right)^2 \ll 1, \tag{61}$$

$$\eta = \frac{V'' \mp 2H^2}{3H^2} \ll 1, \tag{62}$$

$$\delta = -\frac{V_i'B_i}{3NV} \mp \frac{2B^2}{9m_{pl}^2}\frac{1}{(1 - B^2/6m_{pl}^2)} \ll 1. \tag{63}$$

In small-field potential model which means $B^2 \ll m_{pl}^2$, the second term in the slow-roll parameters ϵ, δ would be much smaller than 1. But for large-field potential model where

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$B^2 \geq m_{pl}^2$, we could not get such result. According to the above argument, it seems easier to satisfy slow-roll conditions in the small-field potential model. And Eq. (62) gives $V'' \simeq \mp 2H^2$ which is hard to achieve, at least needing a certain level of tuning.

As for the rapid-roll conditions in the minimal coupling case, we can obtain

$$\epsilon_c = \pm \frac{m_{pl}^2 V_i'^2}{2NV^2} \ll 1, \tag{64}$$

$$\eta_c = \frac{m_{pl}^2 V''}{NV} \pm \frac{(c+2)B_i V''}{3V_i'} + \frac{c(c+2)}{3} \ll 1, \tag{65}$$

$$\delta_c = \frac{\mp \epsilon_c}{c+2} \pm \frac{B_i V_i'}{6V} \ll 1. \tag{66}$$

From Eq. (64), we can see that $\epsilon_c \ll 1$ can always be satisfied as far as N is large enough. And Eq. (66) requires $B_i V_i'/(6V) \ll 1$, which could be satisfied easier in the small-field model than the large-field model as in the situation of slow-roll conditions. Then $\eta_c \simeq 0$ requires

$$c = -(1 + \frac{V''B_i}{2V_i'}) \pm^* \sqrt{(1 - \frac{V''B_i}{2V_i'})^2 - 3\frac{m_{pl}^2 V''}{NV}}. \tag{67}$$

5.2 Example

From the view of gravitational waves, it is interesting that Ref. [8, 29] pointed out only the small field models are feasible. In the following discussions, we will take two examples of potential to discuss the feasible models explicitly from the view of slow-rolling and rapid-rolling. One is the large-field model (chaotic potential $V = m^2 B^2/2$) and the other is the small-field model ($V = V_0 - m^2 B^2/2$ for the positive kinetic energy term case and $V = V_0 + m^2 B^2/2$ for the negative kinetic term case). More definite expressions are present in Tab.(2).

In the large-field potential $V = m^2 B^2/2$, for the slow-roll conditions, it is impossible for the parameter η to be satisfied in the positive kinetic energy case, and in the negative kinetic energy case only when $m^2 \simeq 2H^2$ it could be satisfied. But when $B^2/m_{pl}^2 \gg 1$, we can find that the first term in the brackets of Eq. (63) is so large that it is not suitable for the slow-roll conditions. When $B^2/m_{pl}^2 \simeq 1$ this problem is more serious since both Eqs. (61) and (63) could not be satisfied simultaneously. So the chaotic potential is not proper for the slow-roll conditions. With reference to the rapid-roll conditions, the second term in Eq. (66) which is $1/3$ can exclude the model.

For the small-field potential, as $B^2 \ll m_{pl}^2$ and $m^2 B^2/2 \ll V_0$, the slow-roll conditions that $\epsilon, \delta \ll 1$ can be satisfied. But for the η parameter, the problem still exists that m^2 should be at order of $\mathcal{O}(2H^2)$. Fortunately in the rapid-roll conditions, as $m^2 B^2/2 \ll V$, ϵ_c, η_c and δ_c can be fully satisfied with $c = -1$.

Table 2: Small-field and large-field potential for slow-roll and rapid-roll inflation.

Potentials	Expression	Slow-roll	Rapid-roll
Small-field	$V = V_0 - m^2 B^2/2$ (Positive kinetic energy)	$m^2 \sim \mathcal{O}(2H^2)$	Proper
	$V = V_0 + m^2 B^2/2$ (Negative kinetic energy)	$m^2 \sim \mathcal{O}(2H^2)$	Proper
Large-field	$V = m^2 B^2/2$ (Chaotic potential)	Not proper	Not proper

6   Summary

In the above discussions, we give out the exact forms of the slow-roll and rapid-roll conditions in the space-like vector field inflation. After the assumptions in "cosmic triad" and "N-flation" vector field scenarios being set up, we could see the forms of the minimally coupled vector fields are similar to the non-minimally coupled scalar fields. Because the increasing number of the slow-roll parameters (there is an additional parameter δ) and the fine-tuning model parameters, it is natural that the slow-roll conditions in the vector field scenarios are stricter than the scalar field scenario. When $F = 0$, an special example in the non-minimal coupling case, the slow-roll conditions in the vector field are nearly the same as the minimally coupled scalar field. However, in the minimally coupled vector field case, the slow-roll conditions is much more probable to realize in small-field models rather than in large-field models. And the rapid-roll inflation, as a new model of fast-roll inflation, requires much looser conditions compared with slow-roll inflation, particularly the constraint on η as the examples show. Nevertheless, the positive and negative kinetic energy cases in this model have a lot of similar behaviors. This subject should be further investigated especially from the physical model building aspect.

Acknowledgements

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